HEEGAARD SPLITTINGS OF SEIFERT FIBERED SPACES WITH BOUNDARY

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ABSTRACT. We give the classification theorem for Heegaard splittings of fiberwise orientable Seifert fibered spaces with nonempty boundary. A thin position argument yields a reducibility result which, by induction, shows that all Heegaard splittings of such manifolds are vertical in the sense of Lustig-Moriah. Algebraic arguments allow a classification of the vertical Heegaard splittings.

1. INTRODUCTION

Seifert manifolds were the first 3-manifolds known to possess distinct Heegaard splittings. In [8] Moriah and later in [1] Boileau, Collins and Zieschang described distinct Heegaard splittings of Seifert manifolds which fiber over S^2 and possess three exceptional fibers. In particular, they described two structurally different types of Heegaard splittings for these manifolds: vertical Heegaard splittings and horizontal Heegaard splittings. We here extend the notion of a vertical Heegaard splitting to arbitrary Seifert manifolds and prove the following:

Theorem 4.2. *An irreducible Heegaard splitting of a fiberwise orientable Seifert manifold with nonempty boundary is vertical.*

In most cases, Theorem 4.2 establishes the complete classification of Heegaard splittings for the manifolds in question. Since their vertical Heegaard splittings may be classified using work of Lustig [6] and of Lustig-Moriah [7].

Theorem 5.1. *Let M be a fiberwise orientable Seifert fibered space with nonempty boundary and with Seifert invariants $\{g, e; \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k\}$. Let $\beta = \{i \mid \beta_i \neq \pm 1 \pmod{\alpha_i}; i = 1, \ldots, k\}$. Then the two vertical Heegaard splittings $F(i_1, \ldots, i_n; j_1, \ldots, j_l)$ and $F(\zeta_1, \ldots, \zeta_\nu; j_1, \ldots, j_l)$ are isotopic if and only if $\beta \cap \{i_1, \ldots, i_n\} = \beta \cap \{\zeta_1, \ldots, \zeta_\nu\}$.*

We review definitions of compression bodies and Heegaard splittings (cf. [11]).

Definitions. For M a manifold, denote the boundary of M by ∂M. For S a submanifold of M, denote a closed regular neighborhood of S in M by $N(S)$ and denote an open regular neighborhood of S in M by $\eta(S)$.

A *compression body* W is a 3-manifold which may be constructed by adding 2-handles to a (surface) $\times I$ along a collection of disjoint simple closed curves on (surface) $\times \{0\}$, and capping off any resulting 2-sphere boundary components with 3-balls. The component (surface) $\times \{1\}$ of ∂W is denoted by $\partial_+ W$ and the surface $\partial W - \partial_+ W$, which may or may not be connected, is denoted by $\partial_- W$. If $\partial_- W = \varnothing$, then W is a *handlebody*. If $W = \partial_+ W \times I$, W is called a *trivial compression body*.

A *Heegaard splitting* of M is a pair (W_1, W_2) of compression bodies, such that $M = W_1 \cup W_2$ and $W_1 \cap W_2 = \partial_+ W_1 = \partial_+ W_2 = F$, for some connected closed orientable surface F. This F is called the splitting surface of (W_1, W_2). Two Heegaard splittings of M are considered *equivalent*, if their splitting surfaces are isotopic. A Heegaard splitting is *reducible* if there exists an essential simple closed curve $c \subset F$ that bounds embedded disks in both W_1 and W_2. A Heegaard splitting is *irreducible* if it is not reducible. A *stabilization* of (W_1, W_2) is a Heegaard splitting which is obtained by taking the connected sum of pairs of (M^3, W_1) and a finite number of (S^3, T) summands, where T is the standard unknotted torus in S^3. Note that a Heegaard splitting of an irreducible manifold is reducible if and only if it is a stabilization [3, Haken's theorem].

By Epstein's Theorem, a *Seifert fibered space* is a compact 3-manifold that admits a foliation into circles. We call this foliation a *Seifert fibration* and we call the leaves of the foliation the *fibers* of the Seifert fibration. If we identify each fiber to a point, we obtain a surface P, called the *base* of the Seifert fibration. Note that in general, the natural projection map $p : M \to P$ does not define a fiber bundle in the usual sense, unless we exclude a finite number of points $x_1, \dots, x_k$ of P and the corresponding fibers $e_1, \dots, e_k$ of M. The points $x_1, \dots, x_k$ are called *exceptional points* and the fibers $e_1, \dots, e_k$ are called *exceptional fibers*. We will denote $e_1 \cup \dots \cup e_k$ by E.

Let D_i be a closed regular neighborhood of x_i. Then $V_i = p^{-1}(D_i)$ is a solid torus. This V_i is itself a Seifert fibered space, called a *fibered solid torus*. Note that, in general, the fibration of V_i is not the product fibration (in which fibers would be $\{\text{point}\} \times S^1$).

We will call M a *fiberwise orientable Seifert fibered space*, if both M and P are orientable. To a fiberwise orientable Seifert fibered space M there is associated a set of invariants $\{g, e; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$. The invariant g denotes the genus of P. The invariants (α_i, β_i) are associated to the exceptional fiber e_i. They are determined by the fibered solid torus V_i. Indeed, if V_i is fibered by (p, q) torus knots, then we set $\alpha_i = p$ and require that $0 < \beta_i \leq \alpha_i$ and that $\beta_i \cdot q = 1 \mod p$. The rational Euler number e eliminates any resulting ambiguity. For a definition of e and more detail see [13].

Set $S = \text{closure}(P - (D_1 \cup \dots \cup D_k))$. Then $M - (V_1 \cup \dots \cup V_k)$ is an ordinary circle bundle over the surface S. Since P is orientable and has nonempty boundary, $H^2(S, \bigcup(\partial D_i))$ is trivial. Thus the Euler class of the bundle $M - (V_1 \cup \dots \cup V_k)$ over $P - (D_1 \cup \dots \cup D_k)$ is trivial and hence this bundle is trivial. In particular, if M has no exceptional fibers, then $M = S \times S^1$, and S is a compact orientable surface.

We construct vertical Heegaard splittings for M. This definition extends that of Lustig and Moriah [7] to the case of compact fiberwise orientable Seifert manifolds.
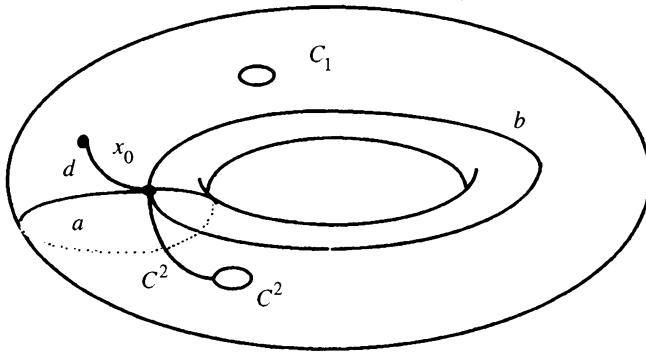
FIGURE 1.1. A collection of arcs defining vertical Heegaard splittings

Roughly speaking, vertical Heegaard splittings are those for which each exceptional fiber appears as the core of a 1-handle in one of the compression bodies. In order to distinguish the various vertical splittings we need to be more precise. Consider a copy of S, together with all the curves here defined, in M. Denote the components of $\partial S - (\partial D_1 \cup \ldots \cup \partial D_k)$ by $C^1, \ldots, C^m$. Let x_0 be a point in $\text{int}(S)$. For $j = 2, \ldots, m$, let c^j be a simple arc in S connecting x_0 to C^j. For $i = 1, \ldots, k$, let d_i be an arc connecting x_0 to ∂D_j. Let $c = c^2 \cup \ldots \cup c^m$ and let $d = d_1 \cup \ldots \cup d_k$. Further, let $a_1, \ b_1, \ \ldots, \ a_g, \ b_g$ be a collection of arcs based at x_0 which cuts S into a disk with punctures corresponding to $C^1, \ \ldots, \ C^m, \ \partial D_1, \ \ldots, \ \partial D_k$. We may assume that all arcs chosen are disjoint except at x_0. See Figure 1.1.

Let $\{i_1, \ldots, i_n\} \subset \{1, \ldots, k\}$ be a collection of distinct indices (for the exceptional fibers) and denote by $\{l_1, \ldots, l_{k-n}\}$ the elements of $\{1, \ldots, k\} - \{i_1, \ldots, i_n\}$. Let $\{j_1, \ldots, j_l\} \subset \{2, \ldots, m\}$ be another collection of distinct indices (for the boundary components) and denote by $\{k_1, \ldots, k_{m-l-1}\}$ the elements of $\{2, \ldots, m\} - \{j_1, \ldots, j_l\}$. Assume that either $n \neq 0$ or $l \neq 0$. Let

$$\Omega(i_1, \ldots, i_n; j_1, \ldots, j_l)$$

be the graph obtained from

$$a_1 \cup b_1 \cup \ldots \cup a_g \cup b_g \cup d \cup \partial D_{l_1} \cup \ldots \cup \partial D_{l_{k-n}} \cup c \cup C^{k_1} \cup \ldots \cup C^{k_{m-l-1}}$$

by modifying

$$d_{l_1} \cup \ldots \cup d_{l_{k-n}} \cup \partial D_{l_1} \cup \ldots \cup \partial D_{l_{k-n}}$$
$$\cup c_{k_1} \cup \ldots \cup c_{k_{m-l-1}} \cup C^{k_1} \cup \ldots \cup C^{k_{m-l-1}}$$

so that they lie in $\text{interior}(S)$. And set

$$W_1(i_1, \ldots, i_n; j_1, \ldots, j_l)$$
$$= N(\Omega(i_1, \ldots, i_n; j_1, \ldots, j_l)$$
$$\cup V_{i_1} \cup \ldots \cup V_{i_n} \cup (C^{j_1} \times S^1) \cup \ldots \cup (C^{j_l} \times S^1)).$$

Set

$$W_2 = \text{closure}(M - W_1).$$

And set

$$F(i_1, \ldots, i_n; j_1, \ldots, j_l)$$
$$= W_1(i_1, \ldots, i_n; j_1, \ldots, j_l) \cap W_2(i_1, \ldots, i_n; j_1, \ldots, j_l)$$
$$= \partial_+ W_1(i_1, \ldots, i_n; j_1, \ldots, j_l)$$
$$= \partial_+ W_2(i_1, \ldots, i_n; j_1, \ldots, j_l).$$

Clearly $W_1(i_1, \ldots, i_n; j_1, \ldots, j_l)$ is a compression body. To see that $W_2(i_1, \ldots, i_n; j_1, \ldots, j_l)$ is a compression body, for instance in the case when $n \neq 0$, isotope x_0 to lie on e_{i_1}, then the arcs a_i, b_i, d_{i_j}, c^{j_i},

$$(\partial(\mathrm{bicollar}(d_{l_j}) \cup \partial D_{l_j})) - (\mathrm{bicollar}(d_{l_j} \cap \partial D_{l_j})),$$

and

$$(\partial(\mathrm{bicollar}(c^{j_i}) \cup C^{j_i})) - (\mathrm{bicollar}(c^{j_i}) \cap C^{j_i})$$

define 2-*handles* in W_2 (e.g., $(a_1 \times S^1) - ((x_0 \times S^1) \cup a_1)$). We can construct W_2 by attaching these 2-*handles* to $F(i_1, \ldots, i_n; j_1, \ldots, j_l)$ and capping off the resulting 2-sphere boundary component with a 3-ball. It follows that

$$(W_1(i_1, \ldots, i_n; j_1, \ldots, j_l), W_2(i_1, \ldots, i_n; j_1, \ldots, j_l))$$

is a Heegaard splitting with splitting surface

$$F(i_1, \ldots, i_n; j_1, \ldots, j_l).$$

Any Heegaard splitting for M obtained in this way is called a *vertical Heegaard splitting*.

When n and l are both 0, $W_1(\varnothing, \varnothing)$ is still a compression body (a handle-body, in fact) in the construction above, but $W_2(\varnothing, \varnothing)$ is not. We may correct this mishap by adjoining $x_0 \times S^1$ to $\Omega(\varnothing, \varnothing)$, but it will be shown in 2.3 that the resulting Heegaard splitting is reducible for the manifolds under consideration. We denote the splitting surface obtained in this modified construction by $F(\varnothing, \varnothing)$.

To see that $F(i_1, \ldots, i_n; j_1, \ldots, j_l)$ is well-defined, apply [12, Theorem 3.1] to $F(i_1, \ldots, i_n; j_1, \ldots, j_l)$ regarded as a Heegaard splitting of $M - \eta(E)$. In the case where $E = \varnothing$, these Heegaard splittings are also called *standard Heegaard splittings of* $P \times S^1$ rel $\{C^{j_1}, \ldots, C^{j_l}\}$.

Note that by forbidding $j_1, \ldots, j_l$ to be 1, we have fixed W_2 as the compression body containing C^1. Also note that the genus of a vertical Heegaard splitting is $2g + k + m - 1$. The genus of $F(\varnothing, \varnothing)$ is $2g + k + m$. When $E = \varnothing$, the standard Heegaard splitting of $P \times S^1 = S \times S^1$ rel $\varnothing$ is in fact irreducible.

The proof of the main theorem relies on an induction argument on the number of exceptional fibers of M. Theorem 1.1 provides the first step in this induction argument.

Theorem 1.1. *For manifolds which are homeomorphic to (compact orientable surface)* $\times$ S^1 *all Heegaard splittings are stabilizations of standard Heegaard splittings.*

Proof. This is [12, Theorem 5.7]. $\square$

Theorem 1.1 has interesting consequences in the setting of Seifert fibered spaces.

Proposition 1.2. *A Heegaard splitting* (W_1, W_2) *of a compact fiberwise orientable Seifert manifold* M *is vertical if and only if, after isotopy,* $(W_1 - \eta(E), W_2 - \eta(E))$ *is a standard Heegaard splitting of* $M - \eta(E)$.

Proof. This follows from the construction. More specifically, denote by C^{m+i_1}, $\ldots$, C^{m+i_n} the components of $\partial(M - \eta(E))$ arising from $e_{i_1}, \ldots, e_{i_n}$, that is, set $C^{m+i_1} = \partial(M - \eta(e_{i_1})) - \partial(M), \ldots, C^{m+i_n} = \partial(M - \eta(e_{i_n})) - \partial(M)$. The splitting surface $F(i_1, \ldots, i_n; j_1, \ldots, j_l)$ of the vertical Heegaard splitting $(W_1(i_1, \ldots, i_n; j_1, \ldots, j_l), W_2(i_1, \ldots, i_n; j_1, \ldots, j_l))$ of M is, after isotopy, the splitting surface of the standard Heegaard splitting of $M - \eta(E)$ rel $\{C^{j_1}, \ldots, C^{j_l}, C^{m+i_1}, \ldots, C^{m+i_n}\}$ and vice versa. $\square$

Proposition 1.3. *If* M *is a Seifert fibered space with a Heegaard splitting formed by the amalgamation (in the sense used in [12]) of vertical Heegaard splittings of Seifert fibered submanifolds along saturated tori, then this Heegaard splitting is vertical.*

Proof. This follows from Proposition 1.2 together with [12, Proposition 2.10]. $\square$

The proof of the Classification Theorem for Heegaard splittings of Seifert manifolds with boundary falls into four parts. In section 2, we define a reducibility property used in the proof of the Main Theorem and give a sufficient criterion for its occurrence. Section 3 provides an elementary technical condition which helps guarantee this criterion. In section 4, we prove the Main Theorem, which says that all irreducible Heegaard splittings of the manifolds under consideration are vertical. In section 5, we provide a classification of the vertical Heegaard splittings via an algebraic invariant.

I wish to thank Martin Scharlemann for numerous helpful discussions.

2. Reducibility

In this section we introduce a reducibility property which enables the inductive step in the proof of the main theorem.

Definition. Let $e \in E$. If, after isotopy, F is a splitting surface for Heegaard splittings both of M and of $M - \eta(e)$, then we say that F is *vertically reducible* at e.

Lemma 2.2 will give sufficient conditions for F to be vertically reducible at e.

Lemma 2.1 (The vacuum lemma). *Suppose* F *is the splitting surface of a Heegaard splitting of the* 3-*manifold* M, D *is an essential disk in* W_1 *and* γ *is a simple closed curve in* F *which intersects* ∂D *once transversely. Then after* γ *is isotoped to lie entirely in* interior(W_1), F *is also a splitting surface of* $M - \eta(\gamma)$. *See Figure* 2.1.

This lemma owes its illustrative name to the following visualization of its proof: The 1-handle dual to the disk D is dragged along γ (γ is vacuumed onto the 1-handle through the attaching disk D of the 1-handle). After this procedure γ lies entirely on the 1-handle; furthermore, γ lies parallel to the
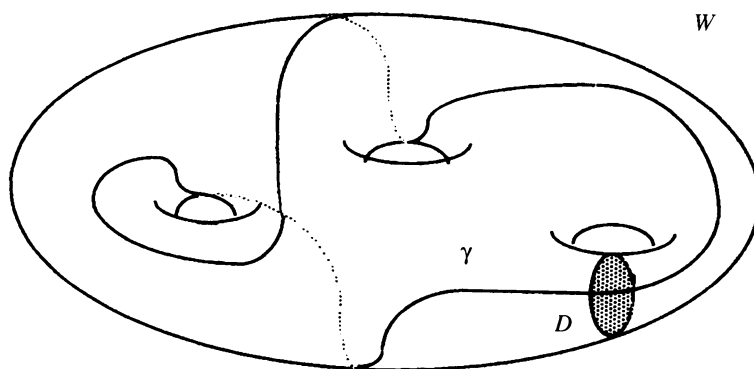
FIGURE 2.1. A curve γ and a disk D in a compression body W.

core of the 1-handle. Drilling out the core of a 1-handle in a compression body produces another compression body.

The following proof of Lemma 2.1 was suggested by Geoffrey Mess during the presentation of a more complicated argument.

Proof. Let $N(\gamma)$ be a collar neighborhood of γ in W_1 and let $N(D)$ be a bicollar of D in W_1. Then $T = N(\gamma) \cup N(D)$ is a solid torus summand of W_1. We may split T from W_1 at the disk

$$D' = \text{closure}[\partial T - \partial_+ W].$$

We may expand the collection $\{D, D'\}$ of boundary reducing disks of W_1 to a defining set of disks Δ of W_1. Set $\Delta(T) = \{D \in \Delta \mid D \subset T\}$. Now $\Delta - \Delta(T)$ is a defining set of disks for $W_1 - T$. Isotope γ to lie in interior(γ) and let $\eta(\gamma)$ be a small regular neighborhood of γ which lies in interior$(N(\gamma))$. Then the collection $(\Delta - \Delta T) \cup \{D'\}$ is a defining set of disks for $W_1 - \eta(\gamma)$. $\square$

Lemma 2.2. *Suppose F is a splitting surface of a Heegaard splitting for M. Suppose further, that $e \in E$ lies in W_1 and is parallel to a simple closed curve γ in F which intersects an essential disk D in W_1 once transversely. Then, after e is isotoped to lie in interior(W_1), F is also a Heegaard splitting for $M - \eta(e)$, so F is vertically reducible at e.*

Proof. Let A be the immersed annulus defined by the isotopy of γ to e. Since e and γ are disjoint and simple, ∂A is embedded. Thus we may assume that A is embedded. Then the lemma follows by replacing $N(\gamma)$ by $N(A)$ in the proof of Lemma 2.1. $\square$

The following lemma will establish the interesting fact that, for the manifolds under consideration, $F(\varnothing, \varnothing)$ is reducible and should hence not be counted among their vertical Heegaard splittings.

Lemma 2.3. *For $E \neq \varnothing$, $F(\varnothing, \varnothing)$ is reducible.*

Proof. Recall the notation used in defining vertical Heegaard splittings. The arc $\delta = (\partial(\text{bicollar}(d_1)) \cup \partial D_1) - (\text{bicollar}(d_1) \cap \partial D_1)$ defines a disk $\tilde{D}$ in W_2 (take $(\delta \times S^1) - (\delta \cup (x_0 \times S^1)))$. Let H be the genus 2 subhandlebody of $W_1(\varnothing, \varnothing)$ defined by $(x_0 \times S^1) \cup \delta$. Let $T = \partial V_1$

$(= \delta \times S^1.)$ Then H may be viewed as (punctured T) $\times I$, with puncture corresponding to $\tilde{D}$.

Let Δ_2 be the compressing disk for W_2 defined by a meridian of V_1. Since Δ_2 is nonseparating in V_1, $\partial \Delta_2$ defines a nonseparating closed curve in $T \subset \partial V_1$. There is a proper arc α in T which intersects this closed curve on T exactly once. This α defines an essential disk Δ_1 in H whose boundary intersects that of Δ_2 exactly once. The pair (Δ_1, Δ_2) shows that F is reducible. $\square$

3. Morse functions and outermost essential saddles

Heegaard splittings correspond to Morse functions. Let (W_1, W_2) be a Heegaard splitting of the 3-manifold M. Then $W_1 = ((\partial_- W_1 \times I) \cup (0\text{-handles})) \cup (1\text{-handles})$, $W_2 = (\partial_+ W_2 \times I) \cup (2\text{-handles}) \cup (3\text{-handles})$, so $W_1 \cup W_2 = ((\partial_- W_1 \times I) \cup (0\text{-handles})) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles})$ gives a handlebody description of M. Excess 0-handles can be canceled with 1-handles and (dually) excess 3-handles with 2-handles, after which there is at most one 0-handle (precisely when W_1 is a handlebody), and at most one 3-handle (precisely when W_2 is a handlebody). This handlebody description can be used to define a Morse function h on M (apply [9, Theorem 3.12] repeatedly). Call h a Morse function induced by (W_1, W_2). Here h can be taken to have singular values $0 < a_1 < \ldots < a_m < b_1 < \ldots < b_n < 1$, with the critical point at level a_i, $i = 1, \ldots, m$, the center of a 1-handle and the critical point at level b_j, $j = 1, \ldots, n$, the center of a 2-handle. For the converse, constructing a Heegaard splitting from a Morse function, see [10, I.3].

Definitions. Let $Q \subset M$ be an immersed compact surface. Let $h : Q \to [0, 1]$ be a Morse function and let r be fixed. An outermost inessential arc α in $h^{-1}(r) \cap Q$ cuts off a disk D; the arc α is called a *high arc* and D a *lower disk* if for all x in $D - \alpha$, $h(x) < r = h(\alpha)$; α is called a *low arc* and D an *upper disk* if for all x in $D - \alpha$, $h(x) > r = h(\alpha)$.

Let Φ_Q be the singular foliation of a compact surface Q by level sets of h. Let σ be a leaf in Φ_Q containing a saddle singularity. By Morse general position, σ contains one critical point x, the saddle point. If x lies in the interior of Q there will be four subarcs of σ, called separatrices, emanating from x. The separatrices may end on ∂Q or may join so that either one pair or two pairs form circles. If a circle formed by separatrices is inessential, then σ is called an *inessential saddle*. Otherwise σ is an *essential saddle*. If x lies on ∂Q, there are two subarcs emanating from x, again called separatrices. In this case σ is called a *half-saddle*. If the two separatrices join to form a circle which is inessential, then σ is called an *inessential half-saddle*. Otherwise σ is called an *essential half-saddle*.

If a point y in Q is a maximum or minimum and lies on ∂Q, then x is called a *half-center*.

Consider $h : Q \to R$ and $C \subset \partial Q$. We say h is *Morse on Q rel C*, if h is Morse on $Q - C$, and near C, h parametrizes a collar $C \times I$ in Q of C, such that h is constant on $C \times \{t\}$. This definition works equally well for manifolds of arbitrary dimension.

Let Q be a compact surface. The saddle σ is an *outermost essential saddle above (below) Q* if
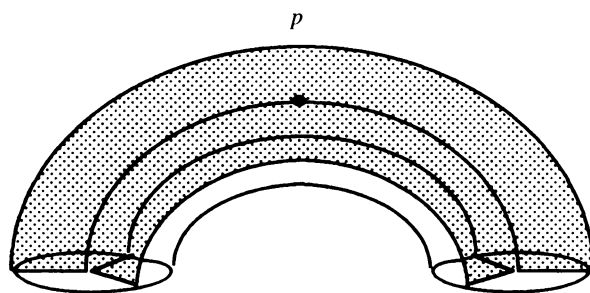
p

Neighborhood of e

FIGURE 3.1

(i) all separatrices of σ end on ∂Q

(ii) σ cuts off two upper (lower) and one lower (upper) disk.

Note: In particular, (ii) implies that all separatrices of σ end on the same component of ∂Q.

Let $e \epsilon E$. Let α be a simple arc in P connecting $x = p(e)$ to ∂P. In the remainder of this section A will denote the saturated annulus $p^{-1}(\alpha)$ which projects to α. Note that A is an immersed but not embedded annulus. We will consider the annulus $\hat{A}$ which is the preimage of this immersion. Let $\partial_1\hat{A}$ be the component of $\partial\hat{A}$ which is mapped into ∂M and let $\partial_2\hat{A} = \partial\hat{A} - \partial_1\hat{A}$. Let $\partial_i A$ be the image of $\partial_i\hat{A}$.

Let h be a Morse function corresponding to the Heegaard splitting defined by the splitting surface F. We may assume that h is Morse on M rel ∂M. The function h restricts to a function on A, which is a Morse function rel $\partial_1 A$ on $A - e$; this function pulls back to a function $\hat{h}$ on $\hat{A}$, which is a Morse function rel $\partial_1\hat{A}$ on $\hat{A} - \partial_2\hat{A}$, but does not extend to all of $\hat{A}$, since critical points of $h|_e$ give rise to two or more critical points on $\partial_2\hat{A}$ occurring at the same level. But if we denote the set of preimages of critical points of $h|_e$ by S, then $\hat{h}|_{\hat{A}-S}$ is a Morse function rel $\partial_1\hat{A}$. We denote by $\Phi_{\hat{A}}$ the foliation of $\hat{A}$ with respect to $\hat{h}$.

Proposition 3.1. *After isotopy of A rel $\partial_1 A$, $\Phi_{\hat{A}}$ contains an outermost essential saddle.*

Lemma 3.2. *After isotopy of A rel $\partial_1 A$, $\Phi_{\hat{A}}$ does not contain any half-saddles.*

Proof. Since h is a Morse function rel ∂M, h takes on a maximal (minimal) value on a connected component C of ∂M and a strictly smaller (greater) value on any point in a collar of C in M. Thus a half-saddle in A cannot have its critical point on C and hence must have its critical point on the exceptional fiber e. Consider a small closed regular neighborhood $N(e)$ of e. Let σ be a half-saddle based at $p \epsilon e$. Let S be a regular neighborhood of p in M consisting of meridian disks of $N(e)$. See Figure 3.1.

Then e has a maximum (minimum) at p, but some components of $A \cap S$ lie above (below) p near p. Here $A \cap S$ consists of squares $A_1, \ldots, A_k$ (k is the index of e) with three sides on ∂S, and one on e. Around p we may use the meridian disks comprising S to isotope $A_1, \ldots, A_k$ to lie entirely below (above) p. The isotopy thus described replaces the half-saddle with a
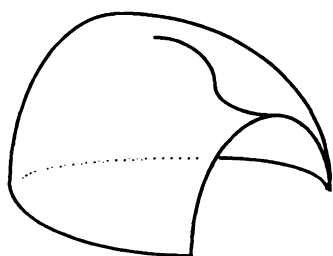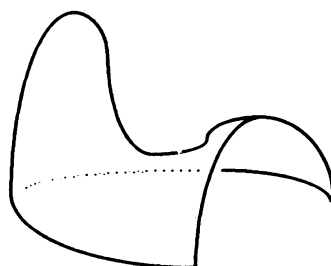
a half-saddle

FIGURE 3.2a
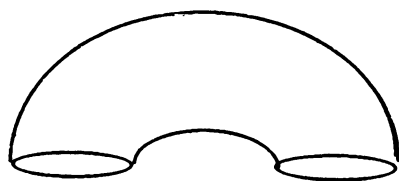

interior saddle point
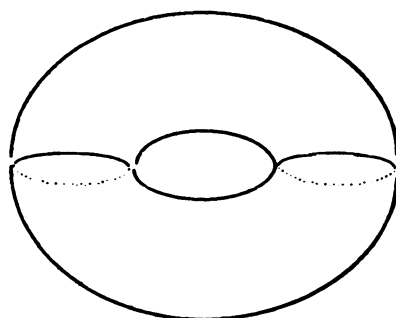and half-center

FIGURE 3.2b

P

FIGURE 3.3a

$\mathscr{M}(P)$

FIGURE 3.3b

half-center and an interior saddle point. It may be performed in an arbitrarily small neighborhood of p. See Figures 3.2a and 3.2b. □

The content of Lemmas 3.3 and 3.4 is similar. Lemma 3.4 will be needed when P is an annulus.

Lemma 3.3. *Suppose Q is a compact surface and $h : Q \to R$ is a Morse function rel ∂Q. Set $\epsilon(Q) = \max\{2 - |\partial Q|, 0\}$ and let Φ be the singular foliation by level sets of h on Q. Then at least $\epsilon(Q) - \chi(Q)$ leaves of Φ are essential saddles.*

Proof. [12, 4.4]. □

Lemma 3.4. *Let P be a planar surface with at least two boundary components. Suppose that Φ_P is the foliation of P with respect to a Morse function rel ∂P such that $h(\partial P)$ is a constant. Then Φ_P contains an essential saddle. See Figures 3.3a and 3.3b.*

Proof. We may assume that $h : P \to [0, 1]$. Without loss of generality, $h(\partial P) = 0$. This implies that $-h$ is also a Morse function rel ∂P on P. Let P' be a copy of P and consider the mirror double $\mathscr{M}(P) = (P \cup P')/ \sim$
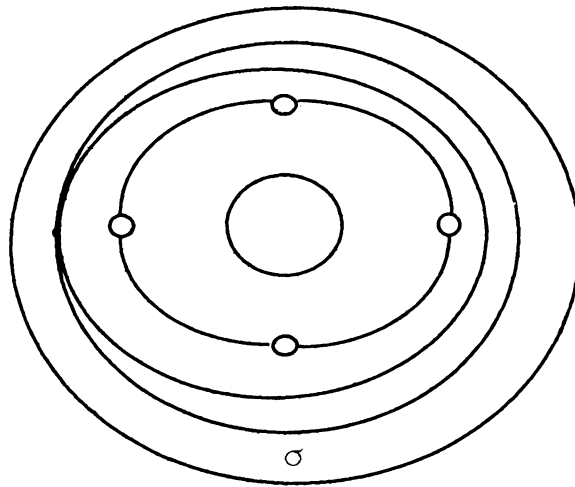
FIGURE 3.4a

of P, where $\sim$ identifies corresponding points in ∂P and $\partial P'$. Now $\mathcal{M}(P)$ is endowed with a Morse function f obtained from h on P and $-h$ on P'. This f is a Morse function on the orientable surface $\mathcal{M}(P)$ of positive genus; and since the Euler characteristic of this surface is less than or equal to 0, $\Phi_{\mathcal{M}(P)}$ contains at least two essential saddles, by Lemma 3.3. Since a bicollar of $[\partial P]$ is foliated by circles, these essential saddles cannot intersect $[\partial P]$, hence lie either entirely in Φ_P or $\Phi_{P'}$. By symmetry, there must be at least one essential saddle in Φ_P. $\square$

Lemma 3.5. *If $\Phi_{\hat{A}}$ contains no half-saddles, then $\Phi_{\hat{A}}$ contains a saddle all of whose separatrices end on $\partial_2 \hat{A}$.*

Proof. Let S be the set of critical points of $h|_e$. Note that $|S| \geq 2$. Denote by $\hat{S}$ the set of critical points in $\hat{A}$ corresponding to $\hat{S}$. Then $|\hat{S}| \geq 4$. Each point in $\hat{S}$ is a half-center. Let $N(\hat{S})$ be a closed regular neighborhood of $\hat{S}$ in $\hat{A}$, such that $\hat{h}$ is constant on each component of $\partial N(\hat{S})$. Set $B = \text{closure}(\hat{A} - N(\hat{S}))$.

Let B' be a copy of B and $(\hat{h}|_{B'})'$ a copy of $\hat{h}|_B$. Consider the double $\mathscr{D}(B) = B \cup B'/ \sim$ of B, where $\sim$ identifies the points in $\partial_2 B = \partial_2 \hat{A} \cap \partial B$ with the corresponding points in $\partial B'$. The function $f : \mathscr{D}(B) \to [0, 1]$, defined by $(\hat{h}|_{B'})'$ on B' and $\hat{h}|_B$ on B is a Morse function rel $\partial \mathscr{D}(B)$ and will be called the *Morse function induced on* $\mathscr{D}(B)$. Here $\mathscr{D}(B)$ is a planar surface with $|\hat{S}| + 2 \geq 6$ boundary components. By Lemma 3.3, $\Phi_{\mathscr{D}(B)}$, the foliation of the $\mathscr{D}(B)$ with respect to f, contains an essential saddle σ.

Since $\Phi_{\hat{A}}$ contains no half-saddles, $|\sigma \cap [\partial_2 B]|$ is even and the saddle point x lies either in B or in B'. We may assume that x lies in B. We may further assume that σ is an outermost essential saddle (which does not a priori mean that σ is an outermost saddle above or below $h^{-1}(\sigma)$).

Case 1: $|\sigma \cap [\partial_2 B]| = 0$ (Figure 3.4a).

Then $\sigma \cap \partial_2 \hat{A} = \varnothing$ and the separatrices combine in pairs to form circles parallel to $\partial_2 \hat{A}$. Let c be the circle closest to $\partial_2 B$. We may consider the surface
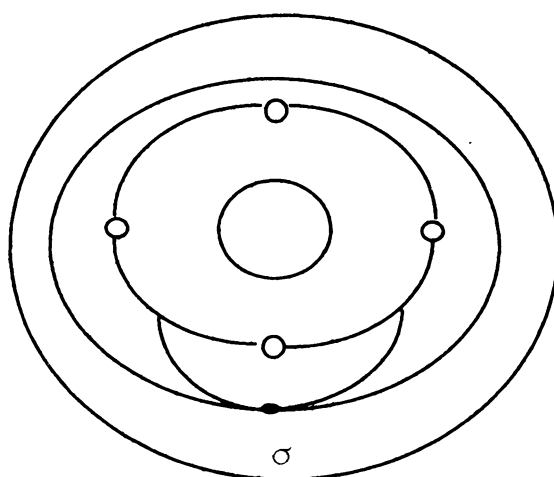
FIGURE 3.4b

$\tilde{B}$ obtained by cutting B along c and taking the component which contains $\partial_2 B$. By the argument above, $\tilde{B}$ contains an essential saddle, contradicting the assumptions.

Case 2: $|\sigma \cap [\partial_2 B]| = 2$ (Figure 3.4b).

Since $\Phi_{\hat{A}}$ contains no half-saddles, the intersections occur in the interior of two of the separatrices. The other pair of separatrices combines to form a circle parallel to $\partial_2 \hat{A}$, since $\hat{h}$ is constant on $\partial_1 \hat{A}$. The leaf σ cuts B into one annulus and two disks D_1, D_2. Here ∂D_1 (respectively ∂D_2) consists of subarcs of σ and of $\partial N(\hat{S})$ together with a collection B_1 (respectively B_2) of subarcs of $\partial_2 B$; here $\partial_2 B = B_1 \cup B_2$.

Claim: Either B_1 or B_2 contains at least four connected components.

We may consider the essential saddle σ in $\hat{A}$. It cuts $\hat{A}$ into one annulus and two disks $\hat{D}_1$ and $\hat{D}_2$, corresponding to D_1 and D_2. Consequently $\partial \hat{D}_1$ ($\partial \hat{D}_2$) consists of subarcs of σ and an arc b_1 (b_2) in $\partial_2 \hat{A}$; where $b_1 \cup b_2 = \partial_2 \hat{A}$. The number of connected components of B_1 (B_2) exceeds the number of critical points on b_1 (b_2) by exactly one. Since e has index at least two, $\hat{h}$ attains its maximum (minimum) at least twice on $\partial_2 \hat{A}$. Note that b_1 and b_2 must each contain an odd number of critical points. So either b_1 or b_2 contains at least three critical points. This proves the claim.

Without loss of generality B_1 contains at least four components. Let l be the level set of $h|_{\hat{A}}$ which contains σ. Cut D_1 along l and consider the component $\tilde{D}_1$ whose boundary contains $\sigma \cap \partial D_1$.

Let $\mathscr{D}(\tilde{D}_1)$ be the double of $\tilde{D}_1$ along $\partial \tilde{D}_1 \cap \partial_1 B$. Then $\mathscr{D}(\tilde{D}_1)$ is a planar surface with at least two boundary components and the induced function on $\mathscr{D}(\tilde{D}_1)$ is a Morse function rel $\partial \mathscr{D}(\tilde{D}_1)$. An argument similar to the one above then shows that $\Phi_{\tilde{D}_1}$, the foliation of $\tilde{D}_1$ with respect to the induced Morse function rel $\partial \tilde{D}_1$, contains an essential saddle, again contradicting the assumptions.

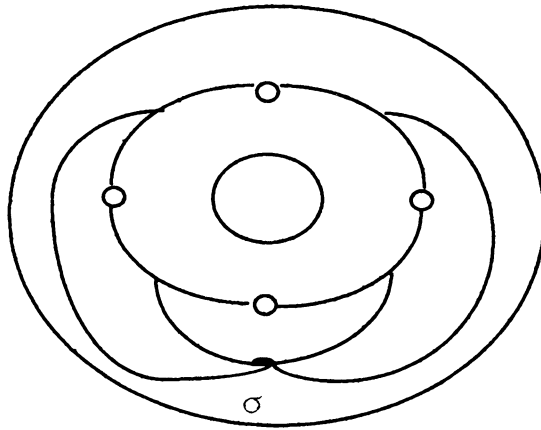*Case* 3: $|\sigma \cap [\partial_2 B]| = 4$ (Figure 3.4c). $\square$

FIGURE 3.4c

Proof. (Proposition 3.1) By Lemma 3.2 and 3.5, $\Phi_{\hat{A}}$ contains, possibly after isotopy of A rel ∂A, a saddle σ all of whose separatrices end on $\partial_2 \hat{A}$. We need only show that either σ cuts off two lower and one upper disk or σ cuts off two upper and one lower disk.

Suppose σ is a saddle in $\Phi_{\hat{A}}$ all of whose separatrices end on $\partial_2 \hat{A}$. If one of the disks D cut off by σ is not an upper or lower disk, then there is a leaf τ in $\Phi_{\hat{A}}$, which occurs at the same level as σ. Since τ occurs at the same level as σ, τ must be a regular leaf.

Case 1: τ is a simple closed curve.

Consider the component $\tilde{A}$ of D cut along τ which contains subarcs of σ. Lemma 3.4 applied to the double $\mathscr{D}(\tilde{A})$ of $\tilde{A}$ along $\partial \tilde{A} - (\sigma \cup \tau)$ shows that $\Phi_{\tilde{A}}$ must contain an essential saddle σ^1.

Case 2: τ is an arc with endpoints on $\partial_2 \hat{A}$.

Let $\tilde{D}$ be the component of D cut along τ which contains subarcs of σ. Lemma 3.4 applied to the double $\mathscr{D}(\tilde{D})$ of $\tilde{D}$ along $\partial \tilde{D} - (\sigma \cup \tau)$ shows that $\Phi_{\tilde{D}}$ must contain an essential saddle σ^1.

An essential saddle in a disk has all separatrices ending on the boundary of that disk. In cases 1 and 2 above, this implies that the essential saddle σ^1 has separatrices ending on $\partial_2 \hat{A}$. We see that if we choose σ to be an outermost saddle with all separatrices ending on $\partial_2 \hat{A}$, then neither case 1 nor case 2 can occur and hence σ must cut off either two upper disks and one lower disk or one upper disk and two lower disks.   □

4. THE MAIN ARGUMENT

Definitions. Let α be a collection of closed curves in the manifold M on which there is defined a Morse function $h : M \to [0, 1]$. After an isotopy of α we may assume that $h|_\alpha$ is a Morse function. Let $r_0, \ldots, r_n$ be the critical values of $h|_\alpha$. Choose $l_1, \ldots, l_n$ so that $r_0 < l_1 < r_1 < \ldots < l_n < r_n$. Let $L_1, \ldots, L_n$ be level surfaces of h corresponding to $l_1, \ldots, l_n$. The number $|L_1 \cap \alpha| + \ldots + |L_n \cap \alpha|$ is called the *width of α with respect to h*. We say
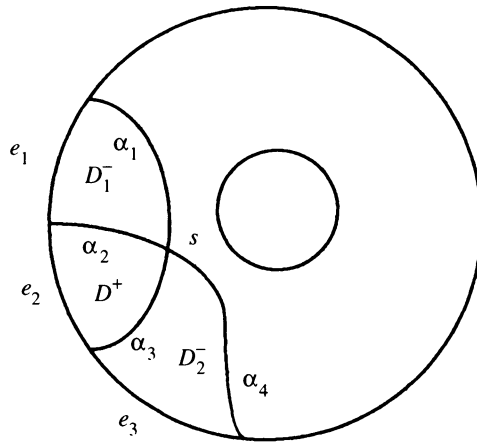
FIGURE 4.1. Labelling of the outermost essential saddle σ

that α *is in thin position with respect to* h if the width of α with respect to h is minimal.

For details see [4, §4] or [5, §1].

Lemma 4.1. *Let* e *be an exceptional fiber in* M. *Then* F *is vertically reducible at* e.

Proof. Let e be an exceptional fiber of M. We may assume that, after a small isotopy, e is disjoint from the the cores of the compression bodies in the Heegaard splitting of M. Thus after this small isotopy, e intersects only level surfaces of h which are isotopic to F. We may also assume that, after isotopy, e is in thin position with respect to h.

Let β be an arc in P connecting ∂P to $x = \pi(e)$. Let A be the saturated annulus which projects to β. Let σ be an outermost essential saddle above or below $\hat{A}$ in $\Phi_{\hat{A}}$ and let L be the level surface of h whose intersection with A induces σ. Without loss of generality, σ is an outermost essential saddle below A, hence cuts off two lower disks, D_1^- and D_2^-, and one upper disk, D^+.

Let s be the critical point of σ. Let α_2 be the subarc of ∂D_1^- shared by ∂D^+, α_1 the subarc of ∂D_1^- connecting s to ∂A which is not in ∂D^+, α_3 the subarc of ∂D_2^- shared by ∂D^+ and α_4 the subarc of ∂D_2^- connecting s to ∂A which is not in ∂D^+. Denote by p_i the endpoint of α_i on e. Further, let e_1 be the subarc of e in ∂D_1^-, e_2 the subarc of e in ∂D^+ and e_3 the subarc of e in ∂D_2^-. See Figure 4.1.

Set $\gamma_1 = \alpha_1 \cup \alpha_3$, $\gamma_2 = \alpha_2 \cup \alpha_4$ and parametrize γ_1, γ_2 as paths. In A, the oriented intersection number $I(\gamma_1, \gamma_2) = \pm 1$. Also in L, the oriented intersection number $I(\gamma_1, \gamma_2) = \pm 1$. (The signs may differ when we look at the oriented intersection number in A versus L.) See Figure 4.2.

Claim 1: The curves γ_1, γ_2 are closed curves based at points p_1, p_2 on e (i.e., $p_1 = p_3$ and $p_2 = p_4$). In particular, $e_1 = e_3$ as subarcs of e.

Since L is both high and low with respect to A, it follows from the proof of [5, Lemma 2.1] or [4, §4] that $p_1 = p_3$ and $p_2 = p_4$. Since e_1, e_3 contain
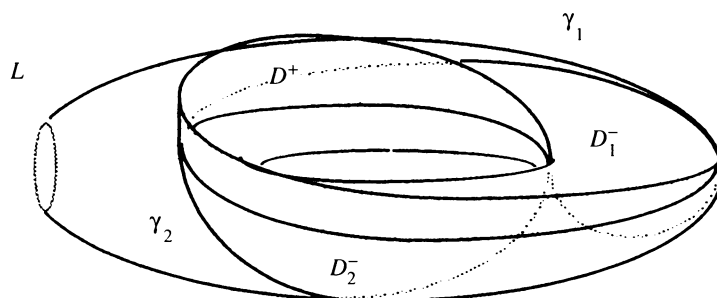
FIGURE 4.2. L, D_1 and D_2 in M
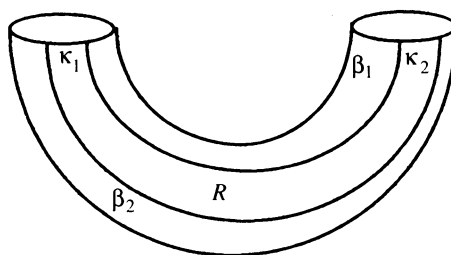
FIGURE 4.3. The arcs β_1, β_3, κ_1 and κ_2

the minimum of e and e_2 contains the maximum of e, the claim follows.

Claim 1 implies that $e_1 \cup e_2 = e$. Hence $D_1^- \cup D^+$ defines an isotopy of γ_1 and e. This fact will be used later.

Now $s = \gamma_1 \cap \gamma_2$. Set $\tilde{A} = A - \eta(e)$, $\tilde{D}_1^- = D_1^- \cap \tilde{A}$, $\tilde{D}_2^- = D_2^- \cap \tilde{A}$, $\tilde{\alpha}_1 = \alpha_1 \cap \tilde{A}$, ..., $\tilde{\alpha}_4 = \alpha_4 \cap \tilde{A}$. Label the endpoint of $\tilde{\alpha}_i$ on $\partial \tilde{A}$ by q_i, and the subarc of $\partial \tilde{A}$ connecting q_i to q_{i+1} by β_i.

Since L is an orientable surface, a copy $\tilde{\gamma}_1$ of γ_1 can be isotoped, in L, off of γ_1 to one side (called the outside, the other side being called the inside; note that since γ_1 may be nonseparating in L, this terminology only makes sense locally). This isotopy can be performed in the complement of γ_2, except near s, where $\tilde{\gamma}_1$ will intersect γ_2 exactly once. So $I(\tilde{\gamma}_1, \gamma_2) = \pm 1$ persists.

Since e intersects L in exactly two points p_1, p_2, $L \cap N(e)$ consists of exactly two disks P_1 and P_2, for $N(e)$ small enough. Now β_1 and β_3 are parallel arcs on $\partial N(e)$, with one endpoint on ∂P_1 and the other on ∂P_2. They both lie below L. On the other hand, β_2 is an arc in $\partial N(e)$, with one endpoint on ∂P_1 and the other on ∂P_2, which lies above L. Note that β_1 and β_3 are parallel in $N(e)$ to e_1, and β_2 is parallel in $N(e)$ to e_2. In particular, in the annulus of $\partial N(e)$ below L, β_1 and β_3 cut off two rectangles with corners q_1, q_2, q_3, q_4. Let R be the rectangle with sides β_1, β_3, κ_1 and κ_2, where κ_1 is chosen to lie on the inside of γ_1 and is hence disjoint from $\tilde{\gamma}_1$. The choice of κ_1 determines κ_2. See Figure 4.3.

Claim 2: $\tilde{D}_1^-$ may be altered slightly, so as to be disjoint from $\tilde{D}_2^-$. Furthermore, this alteration can be made away from $\tilde{\gamma}_1$.

Let $N(s)$ be a small 3-ball neighborhood of s in M missing $\tilde{\gamma}_1$. We replace

the portion of $\tilde{D}_1^-$ which lies in $\text{int}(N(s))$ by a disk in $\partial N(s)$. More specifically, $\partial N(s) \cap \tilde{D}_1^-$ contains an arc τ_1 which together with an arc τ_2 in $L \cap \partial N(s)$ bounds a disk D' in $N(s)$. Replace $\tilde{D}_1^-$ by $\hat{D}_1 = (\tilde{D}_1^- - \text{int}(N(s))) \cup D'$.

Set $D^- = \hat{D}_1^- \cup \tilde{D}_2^- \cup R$. Then since the above alteration was made away from $\tilde{\gamma}_1$, $\partial D^- \cap \tilde{\gamma}_1 = (\gamma_1 \cup \gamma_2) \cap \tilde{\gamma}_1 = \gamma_2 \cap \tilde{\gamma}_1$. Thus $I(\partial D^-, \tilde{\gamma}_1) = \pm 1$. In particular, this implies that ∂D^- is essential and thus that D^- is an essential disk. Now D^- is a lower disk for L such that ∂D^- intersects $\tilde{\gamma}_1$ once transversely. Since L is isotopic to F and e is isotopic to $\tilde{\gamma}_1$, the result now follows from Lemma 2.2. $\square$

Theorem 4.2. *Every irreducible Heegaard splitting of a fiberwise orientable Seifert fibered space with nonempty boundary is vertical.*

Proof. Let M be a Seifert fibered space with nonempty boundary. The proof follows by induction on the number or exceptional fibers in M. Theorem 1.1 establishes the result in the case that M has no exceptional fibers. Now suppose that the result holds for all Seifert fibered spaces with nonempty boundary and at most n exceptional fibers and that M has $n+1$ exceptional fibers. Consider the splitting surface F of an irreducible Heegaard splitting for M. Let e be an exceptional fiber of M. By Lemma 4.1 F is vertically reducible at e. This means that, perhaps after isotopy, F is also the splitting surface of a Heegaard splitting for $M - \eta(e)$. Since $M - \eta(e)$ has only n exceptional fibers F is a vertical Heegaard splitting for $M - \eta(e)$ (irreducibility of a Heegaard splitting in M guarantees irreducibility of the corresponding Heegaard splitting in $M - \eta(e)$). It now follows from the construction that the Heegaard splitting of M with splitting surface F is vertical. $\square$

5. The classification of vertical Heegaard splittings of Seifert fibered spaces after Lustig and Moriah

Given the verticality of irreducible Heegaard splittings of fiberwise orientable Seifert manifolds with boundary, it is natural to ask when two vertical Heegaard splittings are equivalent. This question was answered, in most cases, for closed Seifert fibered spaces in [7]. The content of this section is to tailor Theorem 2.8 in [7] to the class of manifolds under consideration. Lustig and Moriah define an invariant for Heegaard splittings of a closed 3-manifold M, by considering the handlebodies in a Heegaard splitting, and the Nielsen equivalence classes for generating systems of the fundamental group of M which they define. This invariant serves to distinguish vertical Heegaard splittings of Seifert fibered spaces in most cases.

Theorem 5.1. *Let $\beta = \{ i \mid \beta_i \neq \pm 1 \pmod{\alpha_i}; \ i = 1, \ldots, k \}$. Then the two vertical Heegaard splittings $F(i_1, \ldots, i_n; j_1, \ldots, j_l)$ and $F(\zeta_1, \ldots, \zeta_\nu; j_1, \ldots, j_l)$ are isotopic if and only if $\beta \cap \{i_1, \ldots, i_n\} = \beta \cap \{\zeta_1, \ldots, \zeta_\nu\}$.*

Remark 5.2. Consider the Heegaard splittings (W_1, W_2) and (W_1', W_2') of the 3-manifold M with splitting surfaces F and F'. Suppose F and F' are isotopic under an isotopy taking W_1 to W_1'. Then $\partial_- W_1 = \partial_- W_1'$.

Definition. Let $x = \{x_1, \ldots, x_n\}$ be a generating system for a group G. The generating system defines a canonical epimorphism p_x of the free group $F = F(X_1, \ldots, X_n)$ onto G by mapping X_i to x_i. Two generating systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of G are *Nielsen equivalent* if there exists an isomorphism g, for which the following diagram commutes:

$$
\begin{array}{ccc}
F(X_1, \ldots, X_n) & \xrightarrow{\;p_x\;} & G \\[1mm]
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{id}} \\[1mm]
F(Y_1, \ldots, Y_n) & \xrightarrow{\;p_y\;} & G
\end{array}
$$

Remark 5.3. The generators of the fundamental group of a handlebody H in a Heegaard splitting of a closed manifold M define a Nielsen equivalence class $\mathcal{N}(\pi_1(M); H)$ of generators for $\pi_1(M)$. This assignment is unaffected by isotopies of H and hence well defined on an equivalence class of Heegaard splittings. In order to define $\mathcal{N}(\pi_1(M); H)$, H need not be one of the handlebodies in a Heegaard splitting. The only requirement is that its inclusion induce an epimorphism of fundamental groups $i_* : \pi_1(H) \to \pi_1(M)$.

For general facts, such as these, on Nielsen equivalence as pertaining to equivalence classes of Heegaard splittings, see the remarks in [7, Definition 2.8]. (Note the relevance of [15, Theorem 5.8.3].) The two handlebodies in a Heegaard splitting (H_1, H_2) of M may define two different Nielsen equivalence classes $\mathcal{N}(\pi_1(M); H_1)$, $\mathcal{N}(\pi_1(M); H_2)$. For M a merely compact manifold, the assignment of a Nielsen equivalence class of generating systems to a Heegaard splitting of M is less straightforward, since the fundamental group of a compression body is not free.

Remark 5.4. The vertical Heegaard splitting $(W_1(i_1, \ldots, i_n; \; j_1, \ldots, j_l),$ $W_2(i_1, \ldots, i_n; j_1, \ldots, j_l))$ of the fiberwise orientable Seifert fibered space M yields a natural generating system for $\pi_1(M)/\langle h \rangle$, where $\langle h \rangle$ is the subgroup of $\pi_1(M)$ generated by a regular fiber.

For recall the notation used in the introduction to define

$$
(W_1(i_1, \ldots, i_n; j_1, \ldots, j_l), \; W_2(i_1, \ldots, i_n; j_1, \ldots, j_l)).
$$

Parametrize a_1, b_1, $\ldots$, a_g, b_g as paths based at x_0. Let c^j (resp. $\mathfrak{d}_i$) be a parametrization of $c^j \cup C^j$ (resp. $d_i \cup \partial D_i$) as a path based at x_0. And let $\tilde{\mathfrak{d}}_i$ denote the image of e_i, suitably parametrized under $i_* : \pi_1(V_i) \to \pi_1(W_1, x_0)$. Then $\pi_1(W_1)$ is generated by (the homotopy classes of) a_1, b_1, $\ldots$, a_g, b_g, c^2, $\ldots$, c^m, $\tilde{\mathfrak{d}}_{i_1}$, $\ldots$, $\tilde{\mathfrak{d}}_{i_n}$, $\mathfrak{d}_{l_1}$, $\ldots$, $\mathfrak{d}_{l_{k-n}}$ together with l more generators (one per connected component of $\partial_- W_1$) each of which has $\langle h \rangle$ as its image under $i_* : \pi_1(W_1) \to \pi_1(M)$.

Now

$$
\pi_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, c^2, \ldots, c^m, \mathfrak{d}_1, \ldots, \mathfrak{d}_k, h \mid
$$
$$
[a_i, h]\, (i = 1, \ldots, g), \; [b_i, h]\, (i = 1, \ldots, g),
$$
$$
[c^j, h]\, (j = 2, \ldots, m), \; [\mathfrak{d}_l, h]\, (l = 1, \ldots, k),
$$
$$
\mathfrak{d}_l^{\alpha_l} h^{-\beta_l}\, (l = 1, \ldots, k)\rangle.
$$

So

$$\pi_1(M)/\langle h \rangle = \langle a_1, b_1, \ldots, a_g, b_g, \mathfrak{c}^2, \ldots, \mathfrak{c}^m,$$

$$\mathfrak{d}_1, \ldots, \mathfrak{d}_k \ (l = 1, \ldots, k) \mid \mathfrak{d}_l^{\alpha_l} \rangle.$$

To find the natural generating system for $\pi_1(M)/\langle h \rangle$ determined by $W_1(i_1, \ldots, i_n; j_1, \ldots, j_l)$ we thus need only compute the image of $\tilde{\mathfrak{d}}_{i_1}, \ldots, \tilde{\mathfrak{d}}_{i_n}$ under the composition of maps

$$\pi_1(W_1) \xrightarrow{i_*} \pi_1(M) \to \pi_1(M)/\langle h \rangle.$$

This computation is made in [7, Lemma 2.7]. The resulting generators for $\pi_1(M)/\langle h \rangle$ are $\mathfrak{d}_{i_1}^{-\gamma_{i_1}}, \ldots, \mathfrak{d}_{i_n}^{-\gamma_{i_n}}$, where γ_{i_j} is such that if $(\alpha_{i_j}, \beta_{i_j})$ are the invariants associated with e_{i_j}, then for some δ_{i_j}, $\beta_{i_j}\gamma_{i_j} - \alpha_{i_j}\delta_{i_j} = 1$ and $1 \leq \gamma_{i_j} < \alpha_{i_j}$. The natural generating system for $\pi_1(M)$ determined by $W_1(i_1, \ldots, i_n; j_1, \ldots, j_l)$ is thus $\{a_1, b_1, \ldots, a_g, b_g, \mathfrak{c}^2, \ldots, \mathfrak{c}^m, \tilde{\mathfrak{d}}_{i_1}^{-\gamma_{i_1}}, \ldots, \tilde{\mathfrak{d}}_{i_n}^{-\gamma_{i_n}}, \mathfrak{d}_{l_1}, \ldots, \mathfrak{d}_{l_{k-n}}\}$.

Note that this generating system has the same rank as $\pi_1(M)/\langle h \rangle$. Also notice that since $\beta_{i_j} \leq \alpha_{i_j}$ the equality $\beta_{i_j}\gamma_{i_j} - \alpha_{i_j}\delta_{i_j} = 1$ implies $\gamma_{i_j} > 1$.

Remark 5.5. Two generating systems

$$u = x_1^{u_1}, \ldots, x_n^{u_n}, x_{n+1}, \ldots, x_k$$

and

$$v = x_1^{v_1}, \ldots, x_n^{v_n}, x_{n+1}, \ldots, x_k$$

in the free product G of a finite number of cyclic groups,

$$G = \langle x_1, \ldots, x_n, x_{n+1}, \ldots, x_k \mid x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \rangle,$$

are Nielsen equivalent if and only if $u_i = \pm v_i \bmod \alpha_i$ for all $i = 1, \ldots, n$.

This is a consequence of [6, Theorem 1] and Grushko's Theorem.

Lemma 5.6. *If* $\beta_{i_n} = \pm 1 \ (\bmod \alpha_n)$, *then*

$$F(i_1, \ldots, i_n; j_1, \ldots, j_l)$$

and

$$F(i_1, \ldots, i_{n-1}; j_1, \ldots, j_l)$$

are isotopic.

Proof. See [7, Theorem 2.8]. □

Definitions. Let W be a compression body. Then W is obtained from $\partial_- W \times I$ by attaching 1-handles to $\partial_- W \times \{1\}$. Suppose x is a point in $\partial_- W \times \{1\}$ disjoint from the attaching disks of the 1-handles. Then the proper spanning arc

$$\{x\} \times I \subset \partial_- W \times I \subset W$$

is called the *vertical arc based at* $\{x\} \times \{0\}$ *in* $\partial_- W$.

Let $C^1, \ldots, C^n$ be the components of $\partial_- W$ and let $\Gamma = \{\gamma^1, \ldots, \gamma^n\}$ be a collection of vertical arcs such that each γ^j is a vertical arc based at a point in C^j. We call Γ a *vertical collection of spanning arcs for* W. Since $W - \Gamma$ is a handlebody, the inclusion $i : (W - \Gamma) \to W$ defines a Nielsen equivalence

class of generating systems for $\pi_1(W)$ via $i_* : \pi_1(W - \Gamma) \to \pi_1(W)$. We call this class the *Nielsen class of* $\pi_1(W)$ *determined by* Γ.

Two properly isotopic classes Γ and Γ' of vertical spanning arcs for W determine the same Nielsen class of $\pi_1(W)$. Indeed, the isotopy of Γ and Γ' determines an isotopy of the handlebodies $H = W - \Gamma$ and $H = W - \Gamma'$; the Nielsen class of $\pi_1(W)$ determined by Γ is $\mathcal{N}(\pi_1(W); H)$ and the Nielsen class of $\pi_1(W)$ determined by Γ' is $\mathcal{N}(\pi_1(M); H')$, thus they are equivalent by Remark 5.3. In particular, for any two vertical collections of spanning arcs Γ and Γ', the Nielsen class of $\pi_1(W)$ determined by Γ coincides with the Nielsen class of $\pi_1(W)$ determined by Γ', so long as both collections are vertical with respect to the same product structure.

Proof (*Theorem* 5.1). We first prove the following:

Claim: Suppose W and W' are compression bodies in M, with $(W, \partial_- W) \subset (M, \partial M)$ and $(W', \partial_- W') \subset (M, \partial M)$, and such that $i_* : \pi_1(W) \to \pi_1(M)$ and $i'_* : \pi_1(W') \to \pi_1(M)$ are epimorphisms. Further suppose that $G : M \times I \to M$ is a proper isotopy which takes W to W'. Let Γ be a vertical collection of spanning arcs for W and let Γ' be a vertical collection of spanning arcs for W'. Then $\mathcal{N}(\pi_1(M); W - \eta(\Gamma)) = \mathcal{N}(\pi_1(M); W' - \eta(\Gamma'))$.

Denote by $g_t : M \to M$ the function defined by $g_t(x) = G(x, t)$. Loosely speaking we have $\partial_- W \times I \subset W \subset M$ and $\partial_- W' \times I \subset W' \subset M$. But the product structures may differ. To be precise, we have two homeomorphisms $\mathrm{pr} : (\partial_- W \times I) \to (W - (1\text{-handles}))$ and $\mathrm{pr'} : (\partial_- W' \times I) \to (W' - (1\text{-handles}))$ (where $\partial_- W = \partial_- W'$, we will denote the components of $\partial_- W = \partial_- W'$ by $C^1, \ldots, C^m$). Without loss of generality, $\mathrm{pr}|_{\partial_- W \times \{0\}}$ and $\mathrm{pr'}|_{\partial_- W \times \{0\}}$ are projections onto the first coordinate. We must show that $W - \eta(\Gamma)$ and $W' - \eta(\Gamma')$ are isotopic handlebodies. It suffices to show that $g_1(\Gamma)$ and Γ' are properly isotopic in W'. We may consider the connected components of Γ (resp. Γ') individually. It then suffices to show that for each j, $g_1(\gamma^j)$ is properly isotopic to $(\gamma^j)'$ by an isotopy which fixes the complement in M of $\eta(C^j)$.

Suppose γ^j is based at $\{x\} \times \{0\}$ in $C^j \times \{0\} = \mathrm{pr}(C^j \times \{0\})$ and that $(\gamma^j)'$ is based at $\{x'\} \times \{0\}$ in $(C^j) \times \{0\} = \mathrm{pr'}(C^j \times \{0\})$. Without loss of generality, $g_1(\mathrm{pr}(x \times \{1\}))$ is disjoint from the attaching disks for the 1-handles in W'. Let α and β be two curves in C^j such that

(1) $\alpha \cap \beta = x$,
(2) x' lies on α,
(3) $\mathrm{pr}(\alpha \times 1)$, $\mathrm{pr}(\beta \times 1)$ are disjoint from the attaching disks for W and $\mathrm{pr'}(\alpha \times 1)$, $\mathrm{pr'}(\beta \times 1)$ are disjoint from the attaching disks for W'. (See Figure 5.1.)

Set $A = \mathrm{pr}(\alpha \times I)$, $B = \mathrm{pr}(\beta \times I)$, $A' = \mathrm{pr'}(\alpha \times I)$, $\tilde{A} = g_1(A)$ and $\tilde{B} = g_1(B)$. Now $g_1(\gamma^j) = \tilde{A} \cap \tilde{B}$ and thus lies in $\tilde{A}$. On the other hand, $(\gamma^j)'$ lies in A'. Since $g_1|_{C^j \times \{0\}}$ is isotopic to the identity, $g_1(\beta)$ is isotopic to β and hence has the same intersection number, namely 1, with α. Thus the 1-manifold $A' \cap \tilde{B}$ has an odd number of ends on $C^j \times \{0\}$ and hence contains a spanning arc $\tilde{\gamma}^j$. Since $g_1(\gamma^j)$ and $\tilde{\gamma}^j$ are both spanning arcs in $\tilde{B}$ they are properly isotopic; since $\tilde{\gamma}^j$ and $(\gamma^j)'$ are both spanning arcs in A' they are properly isotopic. This establishes the claim.
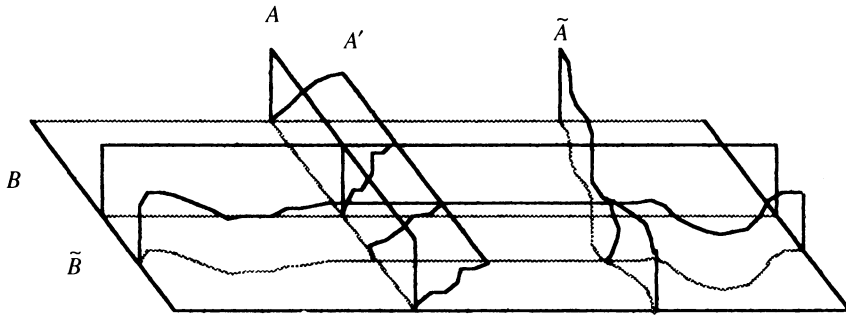
FIGURE 5.1. Partial depiction of the annuli A, A', $\tilde{A}$, B and $\tilde{B}$

We will refer to the Nielsen equivalence class of generating systems for $\pi_1(M)$ provided by W as in the claim by $\mathcal{N}(\pi_1(M); W)$ (i.e., $\mathcal{N}(\pi_1(M); W) = \mathcal{N}(\pi_1(M); W - \Gamma)$, where Γ is a vertical collection of spanning arcs for W). The claim also provides a Nielsen equivalence class of generating systems for $\pi_1(M)/\langle h \rangle$ determined by W, which we denote by $\mathcal{N}(\pi_1(M)/\langle h \rangle; W)$. The generating system calculated in Remark 5.4 is in

$$\mathcal{N}(\pi_1(M)/\langle h \rangle; W(i_1, \ldots, i_n; j_1, \ldots, j_l)).$$

Thus to show that two vertical Heegaard splittings are inequivalent, it suffices to show that

$$\mathcal{N}(\pi_1(M)/\langle h \rangle; W(i_1, \ldots, i_n; j_1, \ldots, j_l))$$
$$\neq \mathcal{N}(\pi_1(M)/\langle h \rangle; W(\zeta_1, \ldots, \zeta_\nu; j_1, \ldots, j_l)).$$

The theorem now follows from Proposition 1.2, Remarks 5.4 and 5.5 and Lemma 5.6. $\square$

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